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A Nonlinear Hille–Yosida Theorem in Banach Spaces*

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Let $(E, |\cdot|)$ be a Banach space. Recall that a (continuous) linear contraction semigroup on E is a family of linear operators $S(t)\colon E \to E$, $0 \leqslant t < \infty$, satisfying the following conditions:

$$S(t+s)x = S(t)\,S(s)x \qquad \text{for all} \quad t, s \geqslant 0 \text{ and } x \in E; \tag{1}$$

$$S(0)x = x \qquad \text{for all } x \in E; \tag{2}$$

$$S(t)x \text{ is continuous in } t \text{ for each } x \text{ in } E; \tag{3}$$

$$\|S(t)\| \leqslant 1 \qquad \text{for each } t \geqslant 0. \tag{4}$$

A linear operator $A\colon D(A) \subset E \to E$ is said to be accretive if

$$|x| \leqslant |x + rAx| \qquad \text{for all } x \in D(A) \text{ and } r > 0. \tag{5}$$

It is called m-accretive if, in addition, the range $R(I + rA) = E$ for all $r > 0$.

In 1948, Hille [6, p. 238] and Yosida [13] proved that there is a bijective correspondence between linear contraction semigroups and linear densely defined m-accretive operators. The operator A can be obtained from the semigroup S by differentiation, $Ax = \lim_{t \to 0+} (x - S(t)x)/t$, and the semigroup can be obtained from A by the exponential formula

$$S(t)x = \lim_{n \to \infty} \left(I + \frac{t}{n}A\right)^{-n} x. \tag{6}$$

A nonlinear (contraction) semigroup on a subset D of E is a family of (nonlinear) operators $S(t)\colon D \to D$, $0 \leqslant t < \infty$, that satisfies (1), (2), (3) (for $x \in D$) and

$$|S(t)x - S(t)y| \leqslant |x - y| \qquad \text{for all } t \geqslant 0 \text{ and } x, y \in D. \tag{7}$$

(In other words, each $S(t)$ is a nonexpansive mapping.)

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A (nonlinear) possibly set-valued operator $A: D(A) \subset E \to 2^E$ is said to be accretive if

$$|x_1 - x_2| \leqslant |x_1 - x_2 + r(y_1 - y_2)| \quad \text{for all } y_i \in Ax_i,\ i = 1, 2, \text{ and } r > 0. \tag{8}$$

As in the linear case, A is called m-accretive if, in addition, $R(I + rA) = E$ for all positive r.

In 1969, Crandall and Pazy [5, Appendix] proved a nonlinear Hille–Yosida theorem in Hilbert space. They established a bijective correspondence between (nonlinear) m-accretive operators and (nonlinear, nonexpansive) semigroups on closed convex subsets of a Hilbert space H. Given an m-accretive operator A, the corresponding semigroup can again be obtained via the exponential formula (6). To obtain A from S, a semigroup on a closed convex subset C of H, Crandall and Pazy first consider the (negative) infinitesimal generator of S defined by

$$A_0 x = \lim_{t \to 0+} (x - S(t)x)/t. \tag{9}$$

(Kōmura [8] has already shown that A_0 is densely defined in C.) Then they extend A_0 to an m-accretive operator A with $D(A) \subset C$.

An m-accretive operator generates a semigroup via the exponential formula in any Banach space [4]. Kōmura's theorem has also been extended outside Hilbert space [1]. On the other hand, the Crandall–Pazy approach is not suited for non-Hilbert spaces because of the extension problem [10].

Thus the question of whether there is a Hille–Yosida theorem in Banach spaces arises. Our purpose in the present article is to offer a (qualified) affirmative answer to this question (cf. [11]).

Since there are semigroups with an empty infinitesimal generator, some restriction has to be imposed on the Banach space E. We shall assume certain differentiability properties of the norms of E and its dual E^*.

Let $U = \{x \in E: |x| = 1\}$. Recall that the norm of E is said to be uniformly Gâteaux differentiable (UG) if $\lim_{t \to 0}(|x + ty| - |x|)/t$ exists for each y in U, uniformly for $x \in U$. It is said to be Fréchet differentiable, (F), if this limit exists for each x in U, uniformly for $y \in U$. A subset C of E is called a nonexpansive retract of E if there exists a retraction of E onto C which is nonexpansive. Finally, let $\mathrm{cl}(D)$ denote the closure of a subset D of E.

THEOREM 1. *Let E be a Banach space. Assume that E is* (UG) *and that E^* is* (F). *If $A \subset E \times E$ is m-accretive, then $C = \mathrm{cl}(D(A))$ is a nonexpansive retract of E and $-A$ generates a semigroup S on C via the exponential formula. Conversely, if S is a semigroup on a nonexpansive retract C of E,* *then there is a unique m-accretive* $A \subset E \times E$ *such that* $\mathrm{cl}(D(A)) = C$ *and* $-A$ *generates* S *via the exponential formula.*

Proof. It is known that E^* is (F) if and only if E is reflexive and strictly convex, and has the following property: If $x_n \rightharpoonup x$ and $|x_n| \to |x|$, then $x_n \to x$. Therefore Crandall's argument on p. 382 of [3] shows that $\mathrm{cl}(D(A))$ is convex. Reference [10, Theorem 2.3] now implies that $\mathrm{cl}(D(A))$ is a nonexpansive retract of E. $-A$ generates a semigroup on C via the exponential formula by [4]. Conversely, let S be a semigroup on a nonexpansive retract C of E, and let $P: E \to C$ be a nonexpansive retraction. Since E is (UG) and reflexive, $J_r x = \lim_{t \to 0+} (I + (r/t)(I - S(t)P))^{-1} x$ exists for each $x \in E$ and $r > 0$ by [11, Theorem 2.1]. Define $A \subset E \times E$ by $A = \bigcup_{r>0} \{[J_r x, (x - J_r x)/r] : x \in E\}$. This A is m-accretive, $\mathrm{cl}(D(A)) = C$, and $-A$ generates S via the exponential formula by the remarks following [11, Theorem 2.1].

Now suppose another m-accretive operator B has the same properties. Let $J_t^B = (I + tB)^{-1}$ be the resolvent of B, and $J: E \to E^*$ the duality map of E. For x in E, set $y_t = (I + (r/t)(I - J_t^B))^{-1} x$. We have

$$((x - J_r^B x)/r - (x - y_t)/r, J(J_r^B x - J_t^B y_t)) \geqslant 0,$$
$$|J_r^B x - J_t^B y_t|^2 \leqslant (y_t - J_t^B y_t, J(J_r^B x - J_t^B y_t)),$$

and

$$|J_r^B x - y_t| \leqslant (2t/r)\, |x - y_t|.$$

Therefore $\lim_{t \to 0+} y_t = J_r^B x$. On the other hand, since $-B$ is assumed to generate S via the exponential formula, the same limit equals $J_r x$ by the uniqueness part of the proof of [11, Theorem 2.1]. Thus the resolvents of A and B are equal, and so are A and B themselves.

Theorem 1 establishes a bijective correspondence between m-accretive operators in $E \times E$ and semigroups on nonexpansive restracts of E. These m-accretive operators also generate the corresponding semigroups via the initial value problem $u'(t) + \mathrm{Au}(t) \ni 0$, $u(0) = x \in D(A)$. In particular, if $-A$ and $-B$ generate the same semigroup, then $A = B$. This provides a positive answer to a question of Kato [7, Question 9.1] (cf. [2]).

For a subset D of E, let $\|D\|$ denote its distance from the origin, and let $D^0 = \{x \in D: |x| = \|D\|\}$. If $A \subset E \times E$, A^0 is defined by $A^0 x = (Ax)^0$.

THEOREM 2. *Let* E *be a Banach space. Assume that* E *is* (UG) *and that* E^* *is* (F). *Then* A_0 *is the* (*negative*) *infinitesimal generator of a semigroup* S *on a nonexpansive retract* C *of* E *if and only if* $A_0 = A^0$ *for an m-accretive* $A \subset E \times E$ *such that* $C = \mathrm{cl}(D(A))$.

Proof. By Theorem 1 all we need to show is that if an m-accretive $A \subset E \times E$ generates a semigroup S on $C = \mathrm{cl}(D(A))$, then $A^0 = A_0$. Since E is smooth and A is maximal accretive, Ax is closed and convex for each $x \in D(A)$. Since E is reflexive and strictly convex, it follows that A^0 is single-valued and $D(A^0) = D(A)$. In fact, A^0 is the (negative) weak infinitesimal generator of S. The result now follows because E^* is (F) and

$$\lim_{t \to 0+} |x - S(t)x|/t = \|Ax\| \qquad \text{for each } x \in D(A).$$

If A and B are m-accretive and $A^0 = B^0$, then the semigroups S_A and S_B generated by $-A$ and $-B$, respectively, solve the same initial value problem by Theorem 2. Therefore $S_A = S_B$ and $A = B$ by Theorem 1. This provides a positive answer to another question of Kato [7, Question 9.2] (cf. [2]). Theorems 1 and 2 include the Hilbert space result of Crandall and Pazy [5, Appendix] (with a different proof) because every closed convex subset of a Hilbert space H is a nonexpansive retract of H. They also provide a bijective correspondence between m-accretive operators $A \subset E \times E$ and semigroups on closed convex subsets of E if E is two-dimensional, strictly convex, and smooth. But in general not every closed convex subset of E is a nonexpensive retract of E [10, Proposition 2.2]. We do have a Hille–Yosida theorem for semigroups on arbitrary closed convex subsets C of E. It is, however, less elegant because by Theorem 1 we can no longer use m-accretive operators. We replace them with accretive operators A such that

$$A = \bigcup_{r>0} \left\{ \left[J_r^A x, \frac{x - J_r^A x}{r} \right] : x \in C \right\}. \tag{10}$$

THEOREM 3. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm, and let S be a semigroup on a closed convex subset C of E. Then there is a unique accretive operator A such that* $\mathrm{cl}(D(A)) = C$, *A satisfies* (10), *and $-A$ generates S via the exponential formula.*

Proof. The existence of such an A again follows from [11, Theorem 2.1]. Its uniqueness can be shown as in the proof of Theorem 1.

In the setting of Theorem 3, suppose that E is also uniformly convex. Then the (negative) infinitesimal generator of S, $A_0 x$, is equal to $\lim_{t \to 0+}(x - J_t x)/t$ [9].

Finally, we remark is passing that other recent nonlinear analogs of classical theorems on linear semigroups (e.g., a complete nonlinear analog of the Trotter–Neveu–Kato theorem and the equivalence between resolvent consistency and convergence for contractive algorithms) can be found in [11] and [12].

Note added in proof. Let E be a Banach space, $A \subset E \times E$ an accretive operator that satisfies the range condition, J_r the resolvent of A, and S the semigroup generated by $-A$. We have recently shown ["A note on the asymptotic behavior of nonlinear semigroups and the range of accretive operators," MRC Technical Summary Report No. 2198, 1981] that if E^* is (F), then the strong $\lim_{t\to 0^+}(x - J_t x)/t$ and the strong $\lim_{t\to 0^+}(x - S(t)x)/t$ exist and are equal for each x in $\hat{D}(A)$ (the generalized domain of A). If, in addition, E is smooth and A is closed, then this common limit equals $A^0 x$. It follows that in the setting of Theorem 3, the (negative) infinitesimal generator of S, $A_0 x$, is equal to $\lim_{t\to 0^+}(x - J_t x)/t$ without any further assumptions. If we let B denote the closure of A, then $A_0 = B^0$, the canonical restriction of B.

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